ANOMALY DETECTION IN TRADING BY MODELING THE UNDERLYING STOCHASTIC PROCESSES

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***Abstract*—We present a new algorithm to estimate a profile of trading and detect instances of market manipulation. The algorithm estimates a profile of the changes to the demand and supply of a stock.**

The algorithm estimates the pattern of trading as a sequence of dependent events with each event contributing to the outcome of the market.

The events are grouped in the order in which they are generated to form sequences. The events segment trading into sequences of a fixed number of events delimited as windows. The time interval of the sequence and length of window in time will vary depending on the activity in the market.

The window controls the resolution of the analysis. The amount of information about the variation lost from averaging.

Each event is an order of a specific type, price and volume in time. The sequence of orders placed is modeled as a Markov process with probability of transitioning from one type of order to another estimated from past observations in the sequence. The sequences can be compared by comparing the transition matrices.

The volume of an order is modeled as being composed of a series of orders of the same type but having a predefined unit volume to make the total effect of the series equivalent to the volume of the original market order.

The algorithm estimates the distribution in the data adapting to changes. It can adapt to new markets and conditions. It operates by comparing trading sessions modeled as sequences to detect those that are outliers. The outliers can be detected onthe variables used to describe the trading and include those patterns that are trading anomalies.

K-Means clustering, principal component analysis and Mahalanobis distance methods of analysis are applied to variables defined on the sequence to identify outliers corresponding to similarity, correlation in types of orders observed during a window, and the distribution in the data respectively. The three kinds of outliers are classified into segments by the degree of overlap in a Venn diagram.

This model of trading relates prices to the sequence in which the orders are placed. The estimated profile of trading can be used to provide information to trade by using past patterns to estimate future trends.

A windowed technique where the performance scales with the number of calculations needed to process a sequence of a fixed number of events.

Keywords-machine learning; survelliance; markov random process; market manipulation; anomaly detection; outlier detection; Mahalanobis distance; principal component analysis; K-Means cluster analysis

I. INTRODUCTION

To ensure confidence in the markets and deliver value to investors and growth in companies and the industry, market surveillance is used to detect market anomalies and manipulations.

The specific manipulations detected grow with experience and use of the algorithm. A database of manipulations and degree of overlap in the detected outliers is a measure of the degree of the outlier and confidence in the alert.

A manipulation can be carried out in several ways and a software configurable alert to detect it using thresholds and parameters to hold the values, would require several values for each parameter that governs the alert making it difficult to optimize such a system.

The manipulations are well defined in a regulatory context. There are also manipulations in the market for which there is no known definition and may not be detected. Such scenarios can be detected by assuming that the manipulations can be detected as market anomalies or abnormal behaviors in trading. Where the anomalies or abnormal behaviors in the trading are the result of a few manipulators while most of the trading is carried out legitimately.

A trading day is segmented into periods of consecutive trading events or sequences. A sequence is a series of orders placed in the stock market. It is formed as a fixed number of events or types of orders and delimited into sequences called windows.

A window is a fixed number of events that represent orders placed at different times and the interval of time taken to form each window varies with the activity in the market. It is used to model trading and present results of the analysis. It is also a period of averaging that can be used to understand variation.

The variation in trading is over time and over events that occur at different times of the day. Longer periods of averaging lose information about the variation but may produce results easier to use and understand.

The variation in the type of orders is modeled as consecutive events delimited by a window. The variation in time can be modeled as the time interval between two events. The two kinds of modeling and analyses can be performed separately and also independently.

Our approach models trading as a random process and is better at detecting anomalies and manipulations than methods which rely on definitions of a particular pattern in the trading. We have defined a set of variables that can detect outliers in them which are also likely to be the result of manipulation.

It makes no assumptions about the distributions of the trading data. We expect it to perform across different markets.

K-Means clustering groups objects using similarity criteria and the cluster(s) most dissimilar considered outliers. Cluster profiles are created to describe the objects in each cluster in terms of the values the variables take in each cluster. They provide information about which variables andcombinations of variables and the values that contribute to the outlier.

Principal component analysis is used to identify outliers as those windows in which the variables are uncorrelated. Cluster analysis can be employed on the principal component scores to further group those data points(windows) that are uncorrelated.

The Mahalanobis distance provides a measure of how far a multidimensional data point lies from the center of its multivariate distribution.

The methods convey different information about how the trading has been done and it is also a way to profile trading. It is also a way to detect different kinds of trading.

II. OVERVIEW OF THE ANOMALY DETECTION ALGORITHM

A. *Modeling sequences of trading events and trading sessions*

Market trading activity can be modeled as a sequence of consecutive events. Each event is a type of order and associated with it time, price and volume. The sequence can be modeled as a sequence of random variables drawn from a first order Markov random process with probability of transition between events estimated in a state transition matrix.

This compact representation of the original sequence can be used to measure randomness in the sequence of events, its long-term behavior (stationary probability distribution), likelihood of observing such a sequence, degree of similarity to other sequences, its composition in terms of its event types. Statistics on the state of the order book help understand and describe the trading modeled as a sequence of events.

The volume of the order is also represented as a sequence of events of a unit volume so that the total numberof unit volume events correspond to the total volume of the order.

B. *Statistical measures estimated on each sequence (window)*

Statistics are estimated on the sequence using the transition matrix. They include

- Entropy of Market Order Sequence - Measure of the randomness in the sequence of order types (sequence of events).
- Stationary Probability Distribution - The long-term convergence behavior of the limiting probabilities of each state indicates the average time spent in each state when the sequence is long.
- Sequence Probability - The likelihood of observing such a sequence.
- Kullback-Leibler Divergence Measure – Measure similarity of the transition matrix to a standard matrix to compare between the distribution of event types.
- Event Type Composition – Distribution of the event types (types of market orders) in the sequence.
- Volume-Event Type Composition – Distribution of the number of unit volume event types in a window.
- Event Inter Arrival Time Entropy – Measures the randomness in the distribution of the inter arrival times of the events in a sequence and the randomness of the events in time.
- Order Book Statistics – Statistics on price, volume and imbalances to describe the state of the order book.

III. FINITE STATE MACHINES AND MARKOV PROCESS

Orders placed with the stock exchange can be modeled as a sequence of events drawn from a finite set of event types. Each type of event corresponds to a particular type of market order. This sequence of events can be modeled as the transitions between the states of a finite state machine. The state transitions are conditional probabilities which are a property of the stochastic process and is estimated for every sequence.

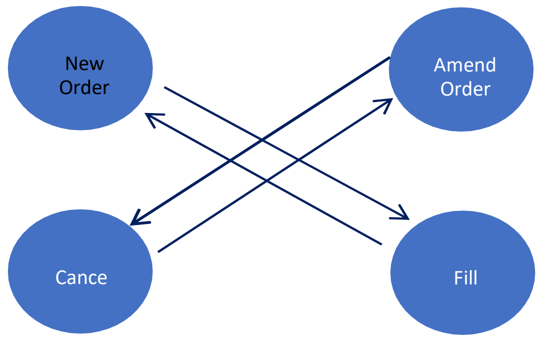

Figure 1. Finite state machine modeling sequence of selected order types

The sequence of orders is modeled as a first order Markov random process where the conditional probability of transitioning from one state to the next, is a state transition matrix [5]. The state transition matrix can be estimated from the sequence of orders by counting the transitions between states. The initial state is a row element and the subsequent state a column. Normalizing by the row sum we obtain a right stochastic matrix.

A. *Markov processes*

A first order Markov process is memoryless. In a sequence of random variables drawn from such a process the distribution of the next random variable in the sequence is dependent only on the present [1].

$$P(X_n \;/\; X_{n-1}X_{n-2}...X_1) = P(X_n \;/\; X_{n-1}) \qquad (1)$$

In this paper, trading is modeled as a first order Markov process. However, higher order Markov models can be used depending on the dependence assumed. The analysis can proceed along similar lines estimating the state transition matrices and variables defined to use with the anomaly detection procedure.

B. *Stationarity*

The probability distribution of a sequence of random variables drawn from a stationary distribution is invariant to a shift in time "*l*".

$$P(X_n X_{n-1} X_{n-2}...X_1) = P(X_{n+l} X_{n+l-1} X_{n+l-2}...X_{1+l}) \qquad (2)$$

The transition matrices estimate the stochastic process which may not be stationary and depends on the period of time over which it is estimated. We compare sequences by comparing transition matrices and use the methods of outlierdetection applied to the transition matrices and statisticsderived from them to find evidence of manipulation.

The degree of stationarity in trading modeled in this way doesn't affect the overall conclusions made with respect to detected outliers.

IV. VARIABLES

A set of "m" variables {F_1,F_2, …, F_m} derived from the stochastic processes are defined on each sequence. They are measured over each window for a sequence of "n" non-overlapping windows as depicted in Figure (1). Each window is defined as a certain number of consecutive eventsor order types.

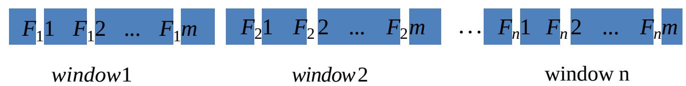

Figure 2. Feature association with each window

The variables are defined to describe and distinguish between patterns in trading. They include differences inmagnitude and similarity, correlation structure and underlying distributions.

The outlier detection is carried out via the methods of Mahalanobis distance, principal component analysis & K-Means clustering.

The transition matrix is a mathematical representation of the finite state machine modeling each sequence. It is the conditional probability of transitioning to a subsequent state X_j from the previous state X_i in a sequence of event

$$P(Xj/Xi) = P(Xi \cap Xj)/P(Xi) \qquad (3)$$

It can be estimated via counts of the sequence of events and order types

$$P(Xj/Xi) = N(Xi \& Xj)/N(Xi) \qquad (4)$$

A state transition matrix is the conditional probability

$$P(i,j) = P(X_n = X_j / X_{n-1} = X_i) \qquad (5)$$

It estimates the conditional probability of making a transition from state X_i at time "n-1" to state X_j at time "n' in a sequence. Where n is the event number in the sequence.

A. *Entropy of sequence*

The entropy rate of a Markov process is defined as a weighted average of the expected values of the logarithm of the conditional distribution. The transition matrix P is the conditional probability of transitioning between states and is estimated over a window by observing a change between two orders in the sequence [2].

$$H(X) = -\sum_{i=1}^{n} \pi_i \sum_{j=1}^{n} P_{i,j} \log(P_{i,j}) \qquad (6)$$

This is a measure of the randomness in the sequence of event types and is a property of the stochastic process.

B. Stationary probabilities

The stationary distribution is obtained as the distribution of states in the limit $\lim_{n \to \infty} P(i, j)^n$

which is the "n" step transition matrix as "n" tends to infinity.

This distribution can also be obtained as the left Eigen Vector of the state transition matrix P.

$$\pi P = \pi \qquad (7)$$

subject to the condition

$$\sum_{i=1}^{n} \pi_i = 1 \qquad (8)$$

There are several methods to solve for π including iteration, diagonalization of the matrix and direct solution of the set of equations or via a least squares solution.

C. Kullback-Leibler divergence criteria [7]

The Kullback-Leibler divergence measure is used to calculate the distance D(T1,T2) between a transition matrix T1 with uniform distribution and the transition matrix T2 estimated for each windowed sequence.

Let T1 and T2 be two transition matrices with s probability distributions each. The asymmetric version of the measure is defined as the distance between two distributions $p1_{ij}$ and $p2_{ij}$ where p_{ij} is the transition probability from state i → j. It is is given by

$$d(p1, p2) = \sum_{i=1}^{s} p1_{ij} \log(p1_{ij}/p2_{ij}) \qquad (9)$$

The symmetric form of the Kullback-Leibler measure is

$$D(T1, T2) = d(p1, p2) + d(p2, p1) \qquad (10)$$

D. Sequence probaility

The probability of a sequence $X_1 X_2 \ldots X_n$ can be calculated by the stationary probability distribution and by theconditional probabilities in the transition matrix

$$P(X_1 X_2 \ldots X_n) = P(X_n / X_{n-1} X_{n-2} \ldots X_1) P(X_{n-1} X_{n-2} \ldots X_1)$$

modeling as a Markov process allows simplification

$$P(X_1 X_2 \ldots X_n) = P(X_1) P(X_2 / X_1) \ldots P(X_n / X_{n-1}) \quad \ldots.. (12)$$

$$P(X_n / X_{n-1} X_{n-2} \ldots X_1) = P(X_n / X_{n-1}) \quad \ldots.. (13)$$

This is also a property of the sequence and of the underlying stochastic process.

E. Event composition of sequence

This variable is a count of each type of order in a window.It is also expressed as a proportion.

F. Volume event composition

The total number of unit volume events in a window, the composition of the window in terms of volume and order type.

G. Entroy of events and inter-arrival time

The time interval between two events depends on how the orders arrive and the market is informed. The distribution of the event inter arrival time is measured on the sequences of events.

We model the inter arrival process as a renewal process where each inter arrival time is independent. The distribution is estimated from historical data.

The autocorrelation of the sequence of inter-arrival times in a window and its entropy is used to measure the degree of correlation and the randomness in the arrival of orders.

The entropy in the types of orders measures the randomness in how the orders appear in a sequence and inform themarket.

H. Order book statistics

The order book provides information about the supply and demand for a security and is used by traders to place orders using what is known about the security.

Movements in the price of the security are determined by orders made at specific volume and price points within the order book. It shows the intent of the broker and perceived demand expressed for a particular security

Statistics describing the state of the order book provides information about how orders are placed to take advantage of the demand and supply for the stock using theinformation available at the time. It provides information about opportunities to trade and manipulate the market.

The maximum and minimum prices at the top of the book, the total depth of the order book, the cumulative volume at particular price points, bid-ask spread and the volume imbalances between buy and sell sides are estimated for each window. They provide information about how prices move in relation to the market for a security.

V. Outlier Detection

Outlier detection is performed using variables that provide information about manipulation in trading patterns and sequences of orders.

A. K-Means cluster analysis

The K-Means algorithm groups objects by the degree of similarity between them. Objects within a cluster are similar and those between clusters dissimilar. The similarity criterion is a distance measure and Euclidean distance is used.

The optimum number of clusters for a cluster analysis is taken to be the knee point of the graph in the number of clusters vs. sum of square error (SSE) in a data set (Fig 3). The SSE is the sum of the within cluster distances to each centroid and is a measure of the goodness of fit of the objects to a cluster.

The algorithm developed to detect this knee point normalizes the number of clusters and the SSE by their respective maxima and draws a tangent to the curve from the end points of the curve. The line bisecting the angle formed when the two tangents meet is extended until it intersects the normalized curve. This point of intersection is the knee point as depicted in Fig 4. The optimum number of clusters is calculated as 0.379*20 = 7.58, rounded to 8.

The cluster membership and the distance to the origin in the original data is depicted in Fig 5. It is observed that cluster 2 are those objects that are most distant from the origin. The outlier cluster is determined as the smallest cluster most distant from the other clusters. The distance to other clusters is the sum of the distances (d_j) between the centroid of the cluster (C_i) to all of the other centroids (C_j).

$$D_i = \sum_{j=1} d_j \quad (14)$$

The outlier cluster is selected as the cluster with the smallest ratio between cluster size and total distance (D) to other clusters (Fig 6). Cluster 2, has 3 objects and the smallest ratio of cluster size to total distance to other clusters. It is a good candidate for an outlier.

A profile of the outlier cluster provides information on which variables are important and how they contribute to the outlier.

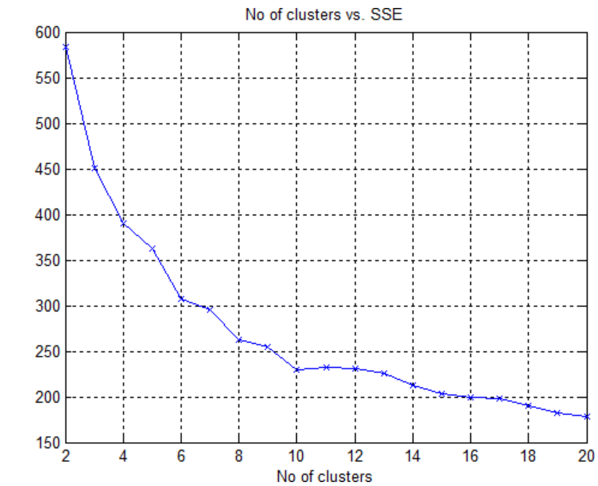

Figure 3. SSE, Goodness of fit - objects to clusters

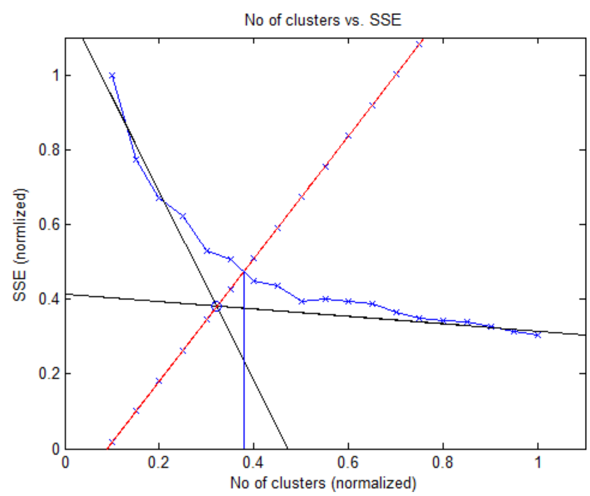

Figure 4. Detecting the knee point

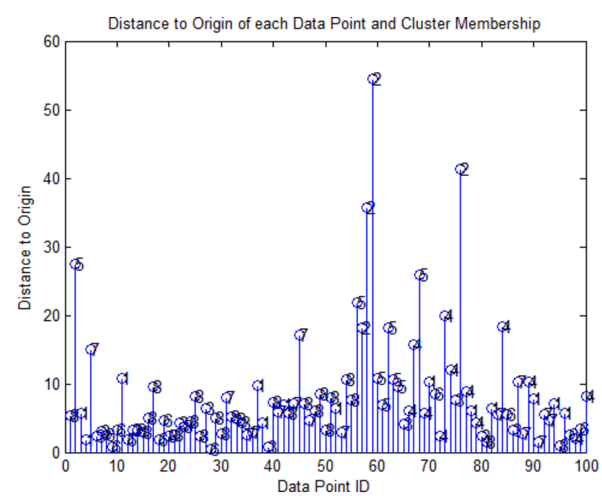

Figure 5. Cluster membership of data

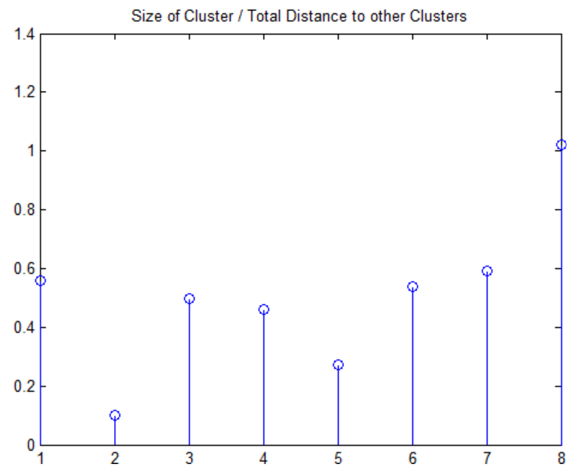

Figure 6. Detecting the outlier cluster

The cluster profile gives the values of the variables that contribute to the similarity in the objects. It is a measure of the similarity in the transition matrices and patterns of trading over time. A measure of the variation in trading overtime and the averaging used to detect instances where the variation is an outlier.

B. *Prinicipal Component Analysis*

The principal components are a set of uncorrelated variables that account for the total variation in the original data. The principal component scores are obtained as linear combinations of the original variables. The transformation is via the Eigen vectors of the correlation matrix of the original data.

When the original variables are standardized the sum of the variances of the original variables is "P" where P is the number of original (standardized) variables. The sum of the Eigen values of the correlation matrix is also P.

The technique is used in the study of the correlation structure in the data. The data points form clusters when the variables are correlated. Variables that are uncorrelated produce outliers.

C. *Mahalanobis Distance*

This measures the distance of a data point from the center of its multivariate distribution. It is sensitive to the spread in the data and is a measure that gives the distance of a data point in the number of standard deviations it lies from the center of its multivariate distribution.

VI. THE ANOMALY DETCTION ALGORITHM

The algorithm can model trading in an instrument or individual traders participating in the market. The trading profile of manipulators would appear as anomalies or outliers when most of the traders trade ethically.

It does not employ rules or thresholds. When the "pattern" is a trader's trading activity, it profiles trading over a period of time estimating a profile of trading.

Legacy systems employ rules, thresholds and triggers to detect a specific type of manipulation. It is difficult to detect specific manipulations and the trading patterns that give rise to them as a single exception to the rules and conditions could invalidate a genuine alert or cause a false alarm.

It is also difficult to define a broad set of conditions to detect a manipulation which can be carried out in more than one way. This will give rise to many combinations for the values of the parameters that govern an alert.

The detection method could look for outliers in an efficient market as there is more than one way to effect a particular kind of manipulation.

A. *Time to effect a manipulation*

The time interval needed to effect a manipulation is uncertain. It depends on the activity in the market. It is assumed that the manipulation is effected through a "sequence" of orders or event types. Where a manipulator or group of manipulators place orders to influence prices with intention to mislead the market.

We study trading behavior over a "period of time" as it takes a "certain amount of time" to effect a manipulation. This interval of time is a window defined in terms of a number of consecutive trading events or manipulative steps.

Windows with overlap can be used at higher cost.

B. *Variables that describe the sequence of events*

The algorithm uses variables that are not dependent on the trading pattern or sequence of events but are derived from it like the accumulated volumes of the order types in a window.

There are also variables that are dependent on the trading patterns or specific sequence of events like the transition matrix used to estimate entropy and long-term distribution.

C. *Modeling a pattern or a sequence of trading events*

Dependence between events in a sequence of events is assumed to model the trading as a sequence of events or orders placed in the market giving a pattern that can be compared with other patterns or sequences.

It is a way of observing how the market for the security changes over time. The order book will represent the perceived demand and supply for the security.

D. *Manipulations that impact the price*

This method models trading and can detect sequences of orders that are evidence of manipulation. Among them the manipulation of the demand and supply for the security.

The cumulative effect of the volume of different types of orders near the top of the book can influence the price.

An "unusual movement of the price" at the instrument level can be correlated with an "unusual imbalance in volumes" measured over a window to provide evidence of a manipulation.

E. *Manipulations that impact liquidy*

Examples of manipulations that influence liquidity include those where traders trade among themselves. The types of orders and accumulated volumes provide evidence of this.

VII. NOVEL CONTRIBUTIONS OF THE ALGORITHM

The algorithm presents a new method of detecting evidence of market manipulation.

- The algorithm models a sequence or orders / pattern of trading as a stochastic process
    - It estimates a profile of trading
    - Doesn't employ thresholds
    - Adapts to the market
- It models orders and the pattern or the sequence in which orders are placed
    - The price, volume, time between orders (order inter arrival time)
    - To detect outliers in the pattern of trading pattern and the attributes of an order
- It compares the sequence of orders or the pattern of trading with other trading patterns
    - To detect those patterns that are very different from the rest
    - To group together patterns that are similar
- The algorithm can profile a traders trading behavior
    - It can model and compare how traders interact with each other.
    - It can compare between trading behaviors
    - To create groups and detect outliers
- New variables can be added to extend the features and functionality of the model
    - New variables to forecast trades / trading
- The algorithm can detect unknown-unknowns
    - It can find new patterns in trading
- The algorithm detects outliers on variables that provide evidence of manipulation
    - It operates at the level of an instrument, broker, trader …
- It uses three methods of outlier detection to detect different kinds of outliers
    - Outliers in similarity
    - Outliers in correlation
    - Outliers in distribution
    - It has means to combine, analyze and view the 3 kinds of outliers

VIII. RESULTS

A Momentum Ignition scenario for a listed security is used to present results on the anomaly detection algorithm.

A strategy to move the price needs to manipulate the demand and supply for the security. A series of orders with an accumulated effect on demand over supply over time can move the price, as presented in Fig 10.

The figures indicate that the outliers in the volume event graphs (Fig 7) correspond with those in entropy (Fig 8) and stationary probability distribution for trade types New Order, Cancel, Amend and Fill (executions), Fig 9. The volume event graph measures the effect of the orders placed and trades on the demand and supply for the security over time.

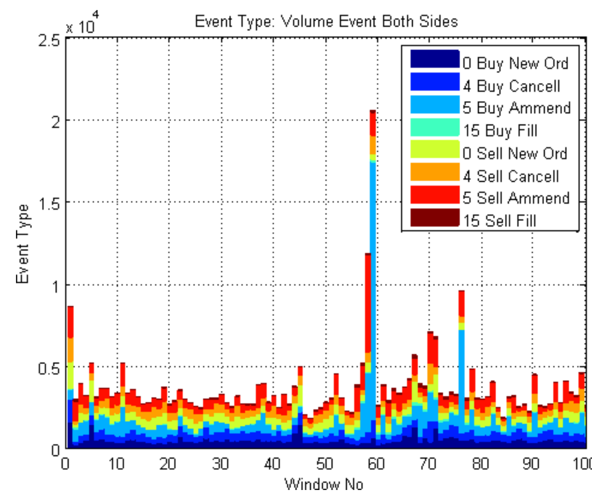

Figure 7. Number of volume events buy and sell side events in a window

We choose a limited set of trade types that make up for more than 95% of all trading events to ensure that the state transition matrices are well populated. The pump and dump manipulation impacts the price at the 58^{th} - 60^{th} window Fig 10, and is detected as an outlier.

Fig 10 presents the maximum and minimum buy and sell prices on each side of the order book over each window.

The entropy plot of Fig 8 detects outliers (windows 58 and 59) in the volume-event sequence when the accumulated volumes measured over the sequences are outliers.

Outliers are detected in the first 3 principal components of the stationary probabilities of both buy and sell sides, Fig 11.

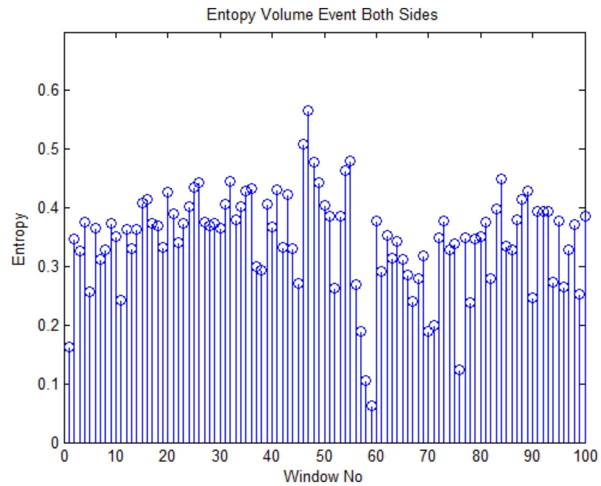

Figure 8. Entropy of buy and sell side unit volume event sequence

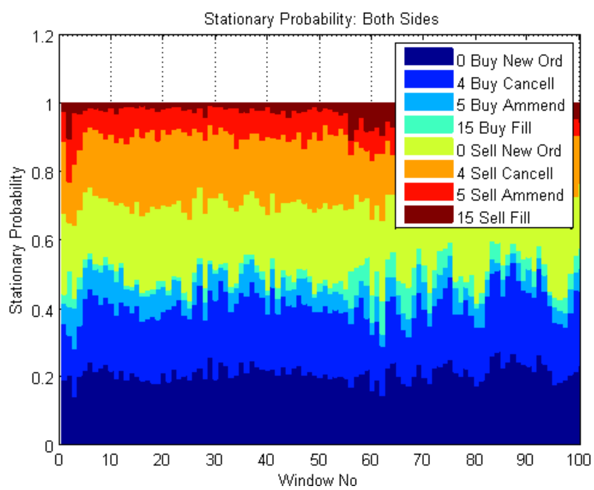

Figure 9. Stationary probability distribution of buy and sell side volume events

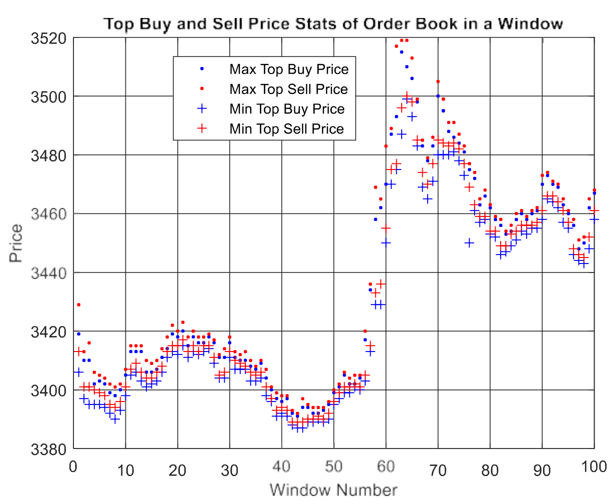

Figure 10. Maximum and minimum price statistics on either side of order book

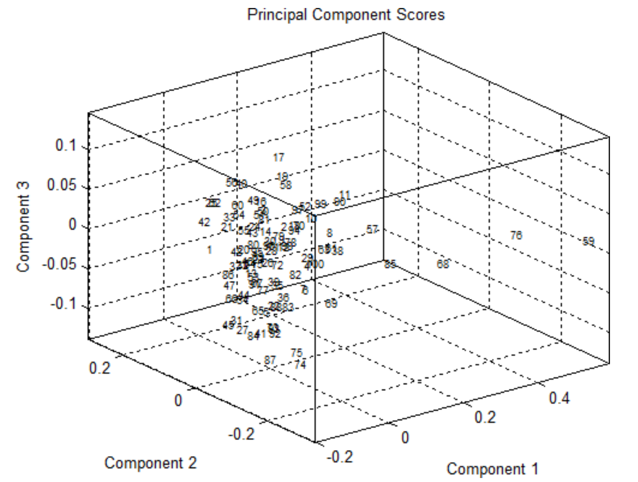

Figure 11. Principal component scores of stationary distribution buy & sell side volume event sequence

The Mahalanobis distance of the stationary probabilities is given in Fig 13. The results are presented for both buy and sell sides together by modeling the orders in the sequence in which they are placed.

Outliers are those windows that are outliers in the distribution of the order types and correspond to the outliers detected via the other methods.

The algorithm can analyze each side separately or together to measure the interactions between the two sides over time. The outliers in this manipulation are different in similarity, correlation of the sequence and distribution in the order types.

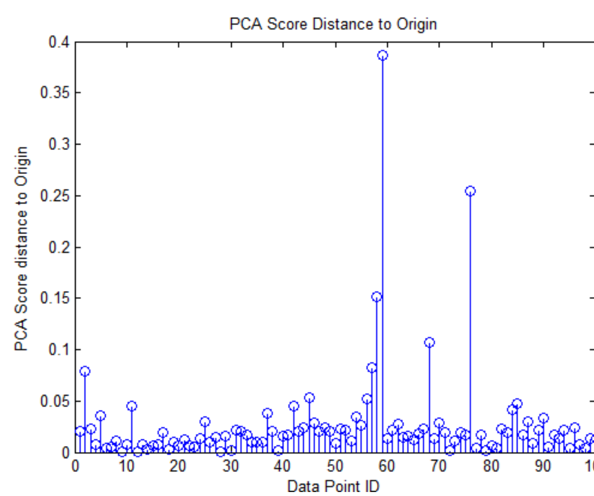

Figure 12. Distance from the origin of principal component scores of stationary distribution of buy & sell side volume event sequence

The distance of the scores from the origin is depicted in Fig 12, where the outliers are dectected. The principal component scores can be clustered via K-Means clustering.

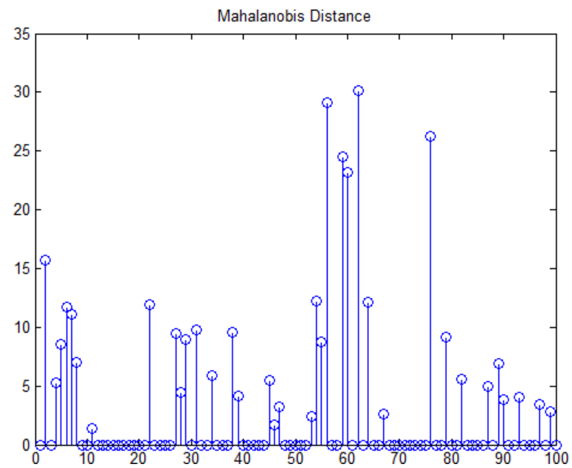

Figure 13. Mahalanobis distance of stationary distribution volume event sequence buy and sell side

It is a way to set up alerts and configure the surveillance system to detect manipulation of specific type. It is about detecting manipulations through the sequence of orders.

The ratio of the accumulated order types from the sequence measured over the window provide information about market trends over time.

IX. CONCLUSION

The anomaly detection algorithm can also be used to trade in the market. It is able to relate orders and prices, the state of the orderbook and price movements. The probability of observing a sequence of orders to the movement in prices.

The trends and risk associated with the prices can be estimated to provide measure of risk in trades and sequence in orders placed.

It is also a way to relate outliers and trades, sequences of orders with price movements and trends in prices and risksin the trends.

The specific kinds of manipulation detected is left as future work and requires examples of manipulated trades to determine the degree of the overlap in the detected outlier.

The Markov model of trading estimated on the sequence of orders provides information about how to detect manipulations.

The algorithm models the sequence of orders placed in the stock market. The variables selected provide information about the sequence, the types of orders and time.

The sequence of orders, counts and proportion of the types of orders provide information about the how state of the order book evolves over time.

A window of events is used to control the resolution of the analysis. It provides the means to average over the variation. The variation due to the variation in the types of orders and the intervals in time at which they are placed. The variation due to the level in activity.

The model leads to the stationary probability and proportion of the orders placed with the market which provides information about the variation in the trading strategy for the security.

The variation about the mean of specific order types and of the totals is a way to detect outliers. They are also evidence of specific kinds of manipulation.

The count and proportions of order types provide information about different kinds of trading and changes to trading.

The transition matrix is an estimate of the pattern and sequence of orders. It also provides information about the randomness in the orders and long-term behavior if the sequence is extended as observed in the past.

Specific kinds of trades and combinations of orders can be detected as a probability of observing a sequence of orders. Outliers provide information about how prevalent it is.

Outliers in these quantities provide information about how trading changes over time. Outliers on variables defined to detect violation of trading rules provide evidence of manipulation.

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